Resonant oscillations in closed tubes

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An investigation is made of the disturbances produced in a closed, gas-filled tube by the oscillations of a piston at one end, when the piston oscillates at near resonant frequencies. Within a well-defined frequency band around each resonant frequency, shock waves appear in the solution; outside this interval the oscillations are continuous, but not purely sinusoidal.

The solution includes the effects of compressive viscosity, and of shear viscosity in the boundary layer at the walls of the tube. For typical laboratory conditions the effect of compressive viscosity is found to be quite small (giving a shock thickness of the order of 10^{-4} in.). The boundary layer effect can be more significant, though the most important modification required of the usual acoustic theory is found to arise from the non-linear terms.

1. Introduction

This paper discusses the disturbances produced in a gas contained in a tube closed at one end ($x = 0$) by a rigid barrier, and at the other ($x = L$) by a vibrating piston. If the displacement of the piston at time t is $l \sin \omega t$ (where $l \ll L$), then acoustic theory says that the particle velocity in the gas is given by

$$u = \frac{l\omega \sin(\omega x/a_0) \cos \omega t}{\sin(\omega L/a_0)}, \tag{1.1}$$

where a_0 is the speed of sound in the undisturbed gas.

The result is clearly invalid at resonant frequencies, where $\sin(\omega L/a_0) = 0$. One might argue that the singularity in the amplitude is in practice eliminated by dissipative effects, such as viscosity and heat conduction. Now it is true that, if the linear theory which leads to equation (1.1) is augmented by some dissipative mechanism, then a finite amplitude is predicted for the oscillations, even at resonant frequencies. The essential modification is to replace $\sin(\omega L/a_0)$ in the denominator of (1.1) by $\{\delta^2 + \sin^2(\omega L/a_0)\}^{\frac{1}{2}}$, where δ is a non-dimensional coefficient of dissipation, depending on the viscosity and heat conduction. However, if δ is small, and this would be so under normal conditions, the result is still inadequate in that large amplitudes are predicted and this is inconsistent with a linear theory.

Moreover, experiments show that, in a narrow frequency band around each resonant frequency, shock waves appear travelling to and fro in the tube being repeatedly reflected from the piston and from the closed end. Near the fundamental frequency one shock wave appears, near the first overtone two shock waves

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and so on. These shocks are accompanied by a noticeable rise in the amplitude of the oscillations compared with the amplitude at non-resonant frequencies, though it can still be small relative to $2\pi a_0/\omega$. Further information is contained in papers by Lettau (1939) and by Saenger & Hudson (1960). Some of the results of experiments by Saenger & Hudson are shown in figures 1 and 2 (plates 1 and 2).

It follows that any attempt to describe the true situation must in some way take account of the non-linear aspects of the problem, a conclusion which has been noted by previous writers. Thus Saenger & Hudson (1960) consider the case in which the piston oscillates at the fundamental frequency by constructing a solution with a built-in shock wave. They start from the assumption that the solution is the sum of a continuous part, which is calculated as a power series in terms of the small parameter l/L, and a discontinuous part satisfying the shock conditions. Compressive viscosity is ignored, but the influence of shear viscosity in the boundary layer near the wall of the tube is considered. In order to retain a one-dimensional model, it is assumed that the effect of friction in the boundary layer can be represented by a term in the momentum equation proportional to the particle velocity. One aspect of their solution is that the amplitude of the oscillations remains finite only by virtue of the action of shear viscosity and heat conduction.

Betchov (1958) also constructs a solution, for resonance at the fundamental frequency, on the assumption that it consists of a continuous and a discontinuous component. Here, however, the main solution stems from the inviscid equations with no dissipative mechanism. The author shows that a disturbance consisting of a shock, on either side of which is an oscillatory flow having a frequency equal to one-half of the fundamental frequency, is consistent with the boundary conditions and approximate relations to be satisfied on the characteristics of the inviscid equations. This solution is interesting in that the amplitude of the oscillation remains small (for small displacements of the piston) albeit of order $(\omega l a_0)^{\frac{1}{2}}$ in the particle velocity rather than (ωl), the magnitude at non-resonant frequencies. Betchov also discusses the modifying effect of wall friction on his solution, assuming, as did Saenger & Hudson, that this effect is equivalent to a body force proportional to the velocity. In both references it is suggested that this effect could in practice modify the solution significantly.

The problem is of sufficient interest to warrant a more detailed discussion of the relative importance of the various mechanisms involved in modifying the acoustic solution. Another aspect of the problem which suggests itself is the question of the transition through near resonant frequencies from a continuous acoustic oscillation to one involving a shock wave.

In the present paper a deductive argument is presented, and shock waves appear as a natural outcome of the solution in a certain well-defined frequency band around each resonant frequency. Except for the interior of shock waves the influence of viscosity and heat conduction would seem to be in the nature of a small correction under normal circumstances, though it would not be too difficult to produce situations where they are significant.

In §2 the appropriate equations for a discussion of the whole problem are derived. However, the arguments used to obtain the basic inviscid solution, which are quite simple and straightforward, can be understood from a reading of §§3 and 4 alone. With the parameters β and δ equated to zero, and with u and $a_0 + a$ interpreted as the particle velocity and the local speed of sound respectively, these two sections are virtually self-contained.

The remaining sections deal with the modification of the inviscid solution by viscosity and heat conduction.

2. Basic equations

For the moment we neglect the influence of the wall friction and discuss the equations for the one-dimensional problem.

Provided that disturbances remain small the most significant terms in the equations of motion will still be linear, though from the previous discussion some account must be taken of the dispersive effects of non-linearity and viscous diffusion. When the disturbance is basically a progressive wave, the appropriate equations have been discussed by Lighthill (1956). The following presentation follows Lighthill with only one minor generalization, which leaves the final equations unaltered but does not restrict their application only to progressive waves.

The equations of motion are written as follows:

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial u}{\partial x} + \left(u \frac{\partial \rho}{\partial x}\right) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial t} + \left(u \frac{\partial u}{\partial x}\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} = \left[\frac{\frac{4}{3}\mu_0 + \mu_{v0}}{\rho_0}\frac{\partial^2 u}{\partial x^2}\right] + \left\{\frac{1}{\rho}\frac{\partial}{\partial x}\left((\tfrac{4}{3}\mu + \mu_v)\frac{\partial u}{\partial x}\right) - \frac{\frac{4}{3}\mu_0 + \mu_{v0}}{\rho_0}\frac{\partial^2 u}{\partial x^2}\right\}, \tag{2.2}$$

$$\rho T \frac{DS}{Dt} = \left\{(\tfrac{4}{3}\mu + \mu_v)\left(\frac{\partial u}{\partial x}\right)^2\right\} + \left[k_0 \frac{\partial^2 T}{\partial x^2}\right] + \left\{\frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) - k_0 \frac{\partial^2 T}{\partial x^2}\right\}. \tag{2.3}$$

The symbols u, ρ, p, T, S denote respectively the velocity, density, pressure, temperature and entropy in the gas, and μ, μ_v, k are respectively the coefficients of viscosity, bulk viscosity and thermal conductivity. The suffix 0 refers to values in the undisturbed gas.

In a wave of velocity amplitude U and time scale ω^{-1}, the appropriate length scale is $a_0\omega^{-1}$. Thus the terms in round brackets in equations (2.1)–(2.3) are of magnitude $O(U/a_0)$ relative to the unbracketed terms, and those in square brackets are of relative magnitude $O(\nu\omega/a_0^2)$, where ν is the kinematic viscosity. All these terms are regarded as significant and are retained, but those in curly brackets are of relative order $(U/a_0)\ (\nu\omega/a_0^2)$ and are neglected.

By (2.3) the entropy changes are merely $O(\nu\omega/a_0^2)$. To calculate this change, retaining only terms of the first order in U/a_0 or $\nu\omega/a_0^2$, equation (2.3) can be simplified to

$$\rho_0 T_0 \frac{\partial S}{\partial t} = k_0 \frac{\partial^2 T}{\partial x^2} \tag{2.4}$$

and, further, variations in T can be replaced by the corresponding variations in particle velocity calculated according to acoustic theory. Thus

$$\frac{\partial S}{\partial t} = \frac{k_0}{\rho_0}\frac{\partial^2}{\partial x^2}\left(\frac{T}{T_0}\right) = \frac{k_0}{\rho_0}\frac{\partial^2}{\partial x^2}\left(\frac{p}{p_0}\right)^{(\gamma-1)/\gamma}$$
$$= \frac{(\gamma-1)k_0}{\gamma\rho_0 p_0}\frac{\partial^2 p}{\partial x^2} = -\frac{(\gamma-1)k_0}{\gamma p_0}\frac{\partial^2 u}{\partial x \partial t}, \tag{2.5}$$

from which
$$S = -\frac{(\gamma-1)\,k_0}{\gamma p_0}\frac{\partial u}{\partial x} \tag{2.6}$$
within the approximation considered. This result can now be used to evaluate
$$\begin{aligned}\frac{1}{\rho}\frac{\partial p}{\partial x} &= \frac{1}{\rho}\left\{\left(\frac{dp}{d\rho}\right)_S\frac{\partial \rho}{\partial x}+\left(\frac{dp}{dS}\right)_\rho\frac{\partial S}{\partial x}\right\} \\ &= a_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\frac{\partial \rho}{\partial x}-\frac{(\gamma-1)\,k_0}{\rho_0 c_p}\frac{\partial^2 u}{\partial x^2},\end{aligned} \tag{2.7}$$
and when this is substituted in (2.2), the resulting equation is
$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+a_0^2\left(\frac{\rho}{\rho_0}\right)^{\gamma-1}\frac{1}{\rho}\frac{\partial \rho}{\partial x} = \delta\frac{\partial^2 u}{\partial x^2}, \tag{2.8}$$
where
$$\delta = \nu_0\left\{\frac{4}{3}+\frac{\mu_{v0}}{\mu_0}+\frac{\gamma-1}{Pr}\right\}, \tag{2.9}$$
with $\nu_0 = \mu_0/\rho_0$ and $Pr = \mu_0 c_p/k_0$.

Equations (2.1) and (2.8) are now in the form given by Lighthill (1956). They are correct as far as terms whose relative order of magnitude is U/a_0 or $\nu\omega/a_0^2$, and contain only the variables u and ρ.

It should be noted that, within the accuracy of this analysis, the energy equation is merely a relation between the entropy changes and the heat conduction within the gas. All irreversible changes are negligible and hence it is consistent to look for solutions in which all the flow variables oscillate about fixed mean values. In the true situation there will of course be irreversible changes, and these must accumulate in a confined region such as a closed tube. But these changes operate on a large time scale, and their effect can be regarded simply as a modification of the undisturbed state to which the suffix 0 applies.

We conclude this section with a discussion of the boundary layer on the wall of the tube, which is assumed to have a small displacement effect on the main flow. To obtain this we first require the solution of the boundary-layer equations, and for the present purpose it is adequate to calculate only the most significant terms in the solution. For small disturbances, the additional terms arising in the boundary layer can be derived from the linearized boundary layer equations. Thus we write
$$u = u_m+u_b, \quad \rho = \rho_m+\rho_b, \tag{2.10}$$
and similarly for the other variables, where u_m, ρ_m are the velocity and density outside the boundary layer. The additional terms in the boundary layer then satisfy
$$\frac{1}{\rho_0}\frac{\partial \rho_b}{\partial t}+\frac{\partial u_b}{\partial x}+\frac{\partial v_b}{\partial n} = 0, \tag{2.11}$$
$$\frac{\partial u_b}{\partial t}+\frac{1}{\rho_0}\frac{\partial p_b}{\partial x} = \nu_0\frac{\partial^2 u_b}{\partial n^2}, \tag{2.12}$$
$$\frac{1}{\gamma-1}\frac{\partial p_b}{\partial t}-\frac{\gamma p_0}{(\gamma-1)\rho_0}\frac{\partial \rho_b}{\partial t} = k_0\frac{\partial^2 T_b}{\partial n^2}, \tag{2.13}$$

where n is a co-ordinate along the inward normal at the wall, and v_b is the velocity component in that direction. The component of the momentum equation normal to the wall is to be interpreted, as usual, to imply that the pressure is unchanged through the boundary layer. Thus $p_b = 0$ and

$$\frac{\partial u_b}{\partial t} = \nu_0 \frac{\partial^2 u_b}{\partial n^2}. \tag{2.14}$$

The solution which satisfies $u_b + u_m = 0$ at the wall is, in operational form,

$$u_b = -u_m \exp\left[-\left(\frac{\sigma}{\nu_0}\right)^{\frac{1}{2}} n\right], \tag{2.15}$$

where σ is the Heaviside operator.

Also, since the pressure is unchanged through the boundary layer, equation (2.13) may be written

$$\frac{\partial T_b}{\partial t} = \frac{\nu_0}{Pr}\frac{\partial^2 T_b}{\partial n^2}. \tag{2.16}$$

To solve (2.16) completely requires a boundary condition at the wall, and this is taken to be $T = T_0$, or $T_b = T_0 - T_m$. Hence

$$T_b = (T_0 - T_m)\exp\left[-\left(\frac{\sigma Pr}{\nu_0}\right)^{\frac{1}{2}} n\right], \tag{2.17}$$

and so

$$\frac{\rho_b}{\rho_0} = \frac{T_m - T_0}{T_0}\exp\left[-\left(\frac{\sigma Pr}{\nu_0}\right)^{\frac{1}{2}} n\right]. \tag{2.18}$$

Equation (2.11) now gives

$$\frac{\partial v_b}{\partial n} = -\frac{1}{T_0}\frac{\partial T_m}{\partial t}\exp\left[-\left(\frac{\sigma Pr}{\nu_0}\right)^{\frac{1}{2}} n\right] + \frac{\partial u_m}{\partial x}\exp\left[-\left(\frac{\sigma}{\nu_0}\right)^{\frac{1}{2}} n\right], \tag{2.19}$$

$$v_b = -\left(\frac{\nu_0}{\sigma Pr}\right)^{\frac{1}{2}}\frac{1}{T_0}\frac{\partial T_m}{\partial t}\left\{1 - \exp\left[-\left(\frac{\sigma Pr}{\nu_0}\right)^{\frac{1}{2}} n\right]\right\} + \left(\frac{\nu_0}{\sigma}\right)^{\frac{1}{2}}\frac{\partial u_m}{\partial x}\left\{1 - \exp\left[-\left(\frac{\sigma}{\nu_0}\right)^{\frac{1}{2}} n\right]\right\}. \tag{2.20}$$

In particular, at the edge of the boundary layer

$$v_b = -\left(\frac{\nu_0}{\sigma Pr}\right)^{\frac{1}{2}}\frac{1}{T_0}\frac{\partial T_m}{\partial t} + \left(\frac{\nu_0}{\sigma}\right)^{\frac{1}{2}}\frac{\partial u_m}{\partial x}, \tag{2.21}$$

and as in (2.5) it is sufficient here to replace the variations in T_m by the corresponding expression in terms of the particle velocity, calculated according to acoustic theory. When this is done, and the operational expression is interpreted, the result is

$$v_b = \left(\frac{\nu_0}{\pi}\right)^{\frac{1}{2}}\left\{1 + \frac{\gamma - 1}{Pr^{\frac{1}{2}}}\right\}\int_0^\infty \frac{\partial u_m(x, t-\xi)}{\partial x}\frac{d\xi}{\xi^{\frac{1}{2}}}, \tag{2.22}$$

where the range of integration will in fact be determined by the range for which the integrand is non-zero.

Finally, we require the effect of the boundary layer on the main flow. To retain a one-dimensional model, the equation of continuity is integrated over the part of the tube cross-section bounded by the edge of the boundary layer. Thus we now write

$$\iint \left\{ \frac{\partial \rho_m}{\partial t} + \frac{\partial(\rho_m u_m)}{\partial x} + \nabla_2 . (\rho_m \mathbf{v}_m) \right\} dA = 0, \tag{2.23}$$

where $\nabla_2 . (\rho_m \mathbf{v}_m)$ is the divergence of the momentum in the plane of the cross-section. Let the same symbols without a suffix denote average values. Then, with the help of the divergence theorem, equation (2.23) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = \frac{1}{A}\int \rho v_b \, ds, \tag{2.24}$$

where the integral on the right-hand side is taken round the edge of the boundary layer and A is the area enclosed by this boundary. Note that the second-order term $\partial\{(\rho_0 - \rho_m) u_m\}/\partial x$ of (2.23) is already $O(U/a_0)$ relative to the linear terms. Hence, when taking the average of this term, the displacement effect can be ignored and the variables regarded as constant over the cross-section within the approximation considered.†

Substitution of (2.22) in (2.24) gives

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = \rho\beta \int_0^\infty u_x(x, t-\xi)\, \xi^{-\frac{1}{2}} \, d\xi, \tag{2.25}$$

where

$$\beta = \frac{2}{R}\left(\frac{\nu_0}{\pi}\right)^{\frac{1}{2}} \left\{1 + \frac{\gamma - 1}{Pr^{\frac{1}{2}}}\right\}, \tag{2.26}$$

and

$$\tfrac{1}{2}R = \frac{\text{cross-sectional area of tube}}{\text{perimeter}}. \tag{2.27}$$

Although R should strictly be defined in terms of the boundary of the outer flow it can in fact be referred to the boundary of the tube itself without loss of accuracy.

The order of magnitude of the right-hand side of (2.25), relative to that of the linear terms on the left-hand side, is $\beta\omega^{-\frac{1}{2}}$. Provided that terms whose relative order of magnitude is less significant than U/a_0, $\delta\omega/a_0^2$ or $\beta\omega^{-\frac{1}{2}}$ are neglected, the momentum equation is not modified and equations (2.8) and (2.25) are adequate within the approximation considered.

If ρ is replaced by

$$a = a_0(\rho/\rho_0)^{\frac{1}{2}(\gamma-1)} - a_0 \tag{2.28}$$

in (2.8) and (2.25), and the resulting equations suitably combined, one finds that

$$\left(\frac{\partial}{\partial t} + a_0 \frac{\partial}{\partial x}\right)\left(u + \frac{2a}{\gamma - 1}\right) = -(u+a)\frac{\partial}{\partial x}\left(u + \frac{2a}{\gamma - 1}\right) + \delta\frac{\partial^2 u}{\partial x^2} + a_0\beta\int_0^\infty u_x(x, t-\xi)\, \xi^{-\frac{1}{2}}\, d\xi, \tag{2.29}$$

† One can also formulate equations for quantities which are averaged over the cross-section of the tube itself, rather than the portion outside the boundary layer. Then the continuity equation is not modified, but the momentum equation is augmented by a body force arising from the shear stress at the wall. The two sets of equations are consistent within the accuracy of the analysis.

$$\left(\frac{\partial}{\partial t}-a_0\frac{\partial}{\partial x}\right)\left(u-\frac{2a}{\gamma-1}\right) = -(u-a)\frac{\partial}{\partial x}\left(u-\frac{2a}{\gamma-1}\right) + \delta\frac{\partial^2 u}{\partial x^2}-a_0\beta\int_0^\infty u_x(x,t-\xi)\,\xi^{-\frac{1}{2}}\,d\xi. \quad (2.30)$$

Here the acoustic terms have been written on the left, and the various additional terms which arise from non-linearity, viscosity and heat conduction appear on the right. Note that the local speed of sound will differ slightly from (a_0+a) because of the entropy variations.

3. The general solution

In the preceding discussion it has been implied that the non-linear terms of the governing equations may have a significant influence in the determination of the solution near resonance. This being so, one might expect that any attempt to obtain a solution by perturbation of the linearized approximation would fail. This is clearly so if by linearized approximation one means expressions such as (1.1) and the corresponding expressions for the other flow variables. However, if the piston oscillates with small amplitude, and if the resulting disturbance in the tube is also of small amplitude, even at resonance, it is reasonable to expect that at least the significant part of the disturbance is still governed by the acoustic equations. The apparent paradox is resolved when one notes that, although the acoustic approximation will in general be the significant part of the disturbance, this will not be so in the neighbourhood of a node of that approximation. At frequencies away from resonance this is not crucial. But if a boundary condition is to be applied in such a neighbourhood, a reliable solution cannot be obtained unless the first approximation is improved, and the extra terms will be all important if the first approximation is locally zero. The following procedure is therefore suggested. A first approximation (u_1, a_1) to (u, a) is taken to be given by

$$u_1+\frac{2}{\gamma-1}a_1 = 2a_0 f_1(t-x/a_0), \quad (3.1)$$

$$u_1-\frac{2}{\gamma-1}a_1 = 2a_0 f_2(t+x/a_0). \quad (3.2)$$

At this stage, f_1 and f_2 are arbitrary functions representing the general solutions of (2.29) and (2.30) when the terms on the right-hand side are neglected. They should be regarded as containing the factors

$$H(t-x/a_0-L/a_0) \quad \text{and} \quad H(t+x/a_0-L/a_0),$$

respectively, where H is the Heaviside unit function. However, we shall be concerned only with the asymptotic behaviour of the solution for large values of the time, and the initial behaviour need be kept in mind only in matters of presentation, where integrals such as those occurring on the right-hand sides of (2.29) and (2.30) are concerned.

A second approximation $(u_1+u_2,\ a_1+a_2)$ is obtained by iteration. The boundary conditions at the ends of the tube are imposed on the second approximation only, and not on the intermediate approximation $(u_1,\ a_1)$.

To carry out the above procedure requires the solutions of the equations

$$\left(\frac{\partial}{\partial t}+a_0\frac{\partial}{\partial x}\right)\left(u_2+\frac{2}{\gamma-1}a_2\right) = 2a_0f_1'(t-x/a_0)\{\tfrac{1}{2}(\gamma+1)f_1(t-x/a_0)$$
$$+\tfrac{1}{2}(3-\gamma)f_2(t+x/a_0)\}+\delta a_0^{-1}\{f_1''(t-x/a_0)+f_2''(t+x/a_0)\}$$
$$-\beta a_0\int_0^\infty\{f_1'(t-x/a_0-\xi)-f_2'(t+x/a_0-\xi)\}\,\xi^{-\frac{1}{2}}\,d\xi, \quad (3.3)$$

$$\left(\frac{\partial}{\partial t}-a_0\frac{\partial}{\partial x}\right)\left(u_2-\frac{2}{\gamma-1}a_2\right) = -2a_0f_2'(t+x/a_0)\{\tfrac{1}{2}(3-\gamma)f_1(t-x/a_0)$$
$$+\tfrac{1}{2}(\gamma+1)f_2(t+x/a_0)\}+\delta a_0^{-1}\{f_1''(t-x/a_0)+f_2''(t+x/a_0)\}$$
$$+\beta a_0\int_0^\infty\{f_1'(t-x/a_0-\xi)-f_2'(t+x/a_0-\xi)\}\,\xi^{-\frac{1}{2}}\,d\xi. \quad (3.4)$$

The complementary functions are of the form (3.1), (3.2) and it is sufficient to obtain particular integrals. Now, for *any* functions $f_1(t-x/a_0)$, $f_2(t+x/a_0)$, we have

$$\left(\frac{\partial}{\partial t}+a_0\frac{\partial}{\partial x}\right)\left\{\frac{x}{a_0}f_1(t-x/a_0)\right\} = f_1(t-x/a_0), \quad (3.5)$$

$$\left(\frac{\partial}{\partial t}+a_0\frac{\partial}{\partial x}\right)\{\tfrac{1}{2}f_1(t-x/a_0)f_2(t+x/a_0)\} = f_1(t-x/a_0)f_2'(t+x/a_0). \quad (3.6)$$

It follows that particular integrals of (3.3) and (3.4) are given by

$$u_2+\frac{2}{\gamma-1}a_2 = f_1'(t-x/a_0)\{(\gamma+1)\,xf_1(t-x/a_0)+\tfrac{1}{2}(3-\gamma)\,a_0F_2(t+x/a_0)\}$$
$$+\delta a_0^{-2}\{xf_1''(t-x/a_0)+\tfrac{1}{2}a_0f_2'(t+x/a_0)\}$$
$$-\beta\int_0^\infty\{xf_1'(t-x/a_0-\xi)-\tfrac{1}{2}a_0f_2(t+x/a_0-\xi)\}\,\xi^{-\frac{1}{2}}\,d\xi, \quad (3.7)$$

$$u_2-\frac{2}{\gamma-1}a_2 = -f_2'(t+x/a_0)\{\tfrac{1}{2}(3-\gamma)\,a_0F_1(t-x/a_0)-(\gamma+1)\,xf_2(t+x/a_0)\}$$
$$+\delta a_0^{-2}\{\tfrac{1}{2}a_0f_1'(t-x/a_0)-xf_2''(t+x/a_0)\}$$
$$+\beta\int_0^\infty\{\tfrac{1}{2}a_0f_1(t-x/a_0-\xi)+xf_2'(t+x/a_0-\xi)\}\,\xi^{-\frac{1}{2}}\,d\xi, \quad (3.8)$$

where

$$F(t) = \int^t f(\xi)\,d\xi. \quad (3.9)$$

The boundary condition $u = 0$ at $x = 0$ is clearly satisfied if

$$f_1 = -f_2 = f \quad \text{(say)}. \quad (3.10)$$

This gives

$$u = u_1+u_2 = a_0f(t-x/a_0)-a_0f(t+x/a_0)$$
$$+\tfrac{1}{2}(\gamma+1)\,x\{f(t-x/a_0)f'(t-x/a_0)+f(t+x/a_0)f'(t+x/a_0)\}$$
$$-\tfrac{1}{4}(3-\gamma)\,a_0\{f'(t-x/a_0)\,F(t+x/a_0)-f'(t+x/a_0)\,F(t-x/a_0)\}$$
$$+\tfrac{1}{2}\delta a_0^{-2}\{xf''(t-x/a_0)+xf''(t+x/a_0)+\tfrac{1}{2}a_0f'(t-x/a_0)-\tfrac{1}{2}a_0f'(t+x/a_0)\}$$
$$-\tfrac{1}{2}\beta\int_0^\infty\{xf'(t-x/a_0-\xi)+xf'(t+x/a_0-\xi)$$
$$-\tfrac{1}{2}a_0f(t-x/a_0-\xi)+\tfrac{1}{2}a_0f(t+x/a_0-\xi)\}\,\xi^{-\frac{1}{2}}\,d\xi. \quad (3.11)$$

This relation for u is now required to satisfy the boundary condition

$$u = l\omega\cos\omega t \quad \text{at} \quad x = L, \tag{3.12}$$

and we are particularly interested in the solution near resonance, that is when

$$\left|\frac{\omega L}{a_0} - N\pi\right| \ll 1 \tag{3.13}$$

for some integer N.

We may therefore write, approximately,

$$\omega L/a_0 - N\pi = \tan(\omega L/a_0), \tag{3.14}$$

a result which will be required presently to simplify the boundary condition on u.

It will be assumed here, and verified later, that the appropriate asymptotic solution for f is periodic with the same frequency as the piston. The integrals which occur in relation (3.11) are also asymptotically periodic with the same frequency. Thus, with the help of (3.14) we have, approximately,

$$f(t+L/a_0) = f(t-L/a_0) + (2/\omega)\tan(\omega L/a_0)f'(t-L/a_0), \tag{3.15}$$

and the boundary condition becomes

$$\begin{aligned} l\omega\cos\omega t = &-2\omega^{-1}a_0\tan(\omega L/a_0)f'(t-L/a_0) \\ &+(\gamma+1)Lf(t-L/a_0)f'(t-L/a_0) + \delta L a_0^{-2}f''(t-L/a_0) \\ &-\beta L\int_0^\infty f'(t-L/a_0-\xi)\,\xi^{-\frac{1}{2}}\,d\xi, \end{aligned} \tag{3.16}$$

provided that terms of less significance than those retained are ignored. In fact, if U/a_0 is regarded as the amplitude of f, the right-hand side of (3.16) is of the form

$$U\{\text{terms whose order of magnitude is } \tan(\omega L/a_0),\ U/a_0,\ \delta\omega/a_0^2 \quad \text{or} \quad \beta\omega^{-\frac{1}{2}}\}.$$

The approximation is consistent throughout provided terms whose order of magnitude is less significant than these are neglected.

The final equation chosen to define f is a simple modification of (3.16). Without loss of accuracy, this equation may be written

$$\begin{aligned} \frac{l\omega\cos\omega t}{(\gamma+1)L\cos(\omega L/a_0)} = &-\frac{2a_0}{(\gamma+1)\omega L}\tan(\omega L/a_0)f'(t) \\ &+f(t)f'(t) + \frac{\delta}{(\gamma+1)a_0^2}f''(t) - \frac{\beta}{\gamma+1}\int_0^\infty f'(t-\xi)\xi^{-\frac{1}{2}}\,d\xi. \end{aligned} \tag{3.17}$$

Equation (3.17) has the advantage that it contains the classical acoustic solution away from resonance. For, when $\tan(\omega L/a_0)$ is not small, f is given essentially by

$$\frac{l\omega\cos\omega t}{(\gamma+1)L\cos(\omega L/a_0)} = -\frac{2a_0}{(\gamma+1)\omega L}\tan(\omega L/a_0)f'(t),$$

or

$$f(t) = -\frac{\omega l\sin\omega t}{2a_0\sin(\omega L/a_0)}, \tag{3.18}$$

and this yields (1.1) when used in (3.1) and (3.2). Thus we may take (3.17) to be the uniformly valid equation for f for all frequencies. It can be integrated once to give

$$\begin{aligned} c + \tfrac{1}{2}\epsilon\cos(\omega t - \tfrac{1}{2}j\pi) = &-\frac{4a_0}{(\gamma+1)\omega L}\tan(\omega L/a_0)f(t) \\ &+f^2(t) + \frac{2\delta}{(\gamma+1)a_0^2}f'(t) - \frac{2\beta}{\gamma+1}\int_0^\infty f(t-\xi)\,\xi^{-\frac{1}{2}}\,d\xi, \end{aligned} \tag{3.19}$$

where
$$\epsilon = \frac{4l}{(\gamma+1)L\,|\cos(\omega L/a_0)|}, \tag{3.20}$$
$$\left.\begin{aligned} j &= 1 \quad \text{if} \quad \cos(\omega L/a_0) > 0 \\ &= -1 \quad \text{if} \quad \cos(\omega L/a_0) < 0 \end{aligned}\right\} \tag{3.21}$$
and c is some constant of integration.

The problem is now reduced to that of solving equation (3.19) for f. At this stage it is worth repeating that the main purpose of this refinement of acoustic theory is simply to ensure that the boundary condition at the piston is properly satisfied. The significant part of the disturbance in the body of the tube is an acoustic oscillation in the sense that it satisfies the wave equation, and it is given by equation (3.1) and (3.2) once the function $f(=f_1=-f_2)$ is known. In most applications this solution will be quite adequate and no attempt will be made to improve upon it in the subsequent analysis.

4. The inviscid solution

As a matter of presentation, the inviscid solution is regarded as the basic solution and treated separately. In subsequent sections the other terms which contribute to the right-hand side of equation (3.19) will be singled out and their modifying effect will be discussed.

When the effects of viscosity and heat conduction are ignored, equation (3.19) may be written
$$(f-2r\epsilon^{\frac{1}{2}}/\pi)^2 = \epsilon(b^2+\cos^2\tau), \tag{4.1}$$
where
$$2\tau = \omega t - \tfrac{1}{2}j\pi, \tag{4.2}$$
$$r = \frac{\pi a_0 \tan(\omega L/a_0)}{(\gamma+1)\,\omega L \epsilon^{\frac{1}{2}}} \tag{4.3}$$
and b is some constant still to be determined.

Now the appropriate function f for the actual problem should have zero mean value. This follows from (3.1) and (3.2) and the fact that both u and a have been defined in such a way that their mean values are zero. The continuous solutions of (4.1) can be made to satisfy this condition provided $|r| > 1$. These solutions are periodic with the same period as the piston oscillations and may be written in the form
$$f = \epsilon^{\frac{1}{2}}\left[\frac{2r}{\pi} \pm (b^2+\cos^2\tau)^{\frac{1}{2}}\right], \tag{4.4}$$
where the question of sign, and the magnitude of b, are determined by the condition that the mean value of f shall be zero. Thus
$$|r| = \int_0^{\frac{1}{2}\pi} (b^2+\cos^2\tau)^{\frac{1}{2}}\,d\tau = (1+b^2)^{\frac{1}{2}} E\{(1+b^2)^{-\frac{1}{2}}\}, \tag{4.5}$$
where E is the complete elliptic integral of the second kind.

For $|r| \gg 1$, (4.4) and (4.5) simplify to
$$b^2 = (4r^2/\pi^2) - \tfrac{1}{2}, \tag{4.6}$$
$$f = -(\pi\epsilon^{\frac{1}{2}}/8r)\cos 2\tau = \frac{-\omega l \sin\omega t}{2a_0 \sin(\omega L/a_0)}, \tag{4.7}$$
which agrees with (3.18) and yields the acoustic solution (1.1).

For $|r| = 1$, $b = 0$ and the expression for f becomes a rectified sine wave, suitably displaced so that the mean value is zero.

For $|r| < 1$, there is no continuous solution of (4.1) with zero mean value. However, one must consider the possibility of a composite solution including discontinuities, particularly since shock waves are suspected. Since there are two possible continuous solutions of (4.1) for any given value of b, such a composite solution can certainly be found having zero mean value. However, if b is non-zero, such solutions would necessarily contain discontinuities both of compression and rarefaction. It is true that acoustic theory shows no preference for one type of discontinuity rather than another, but there is sufficient background of evidence in gas dynamics to suggest that a discontinuity of rarefaction, even within the framework of acoustics, would not be produced by the sinusoidal oscillations of a piston, and surely not in the body of the gas away from the boundaries.

There remains the possibility $b = 0$ (it is clear from (4.1) that b^2 can not be negative). Here the solutions of (4.1) may be written in the form

$$f = \epsilon^{\frac{1}{2}}[(2r/\pi) \pm \cos\tau], \tag{4.8}$$

and a composite solution can now be obtained with zero mean value and with discontinuities of compression only. It is most conveniently defined as a periodic function with the same period as the piston oscillations, namely $2\pi/\omega$ in t or π in τ, and such that

$$f = \epsilon^{\frac{1}{2}}[(2r/\pi) + \cos\tau], \quad \sin^{-1} r < \tau < \pi + \sin^{-1} r, \tag{4.9}$$

where $-\frac{1}{2}\pi \leqslant \sin^{-1} r \leqslant \frac{1}{2}\pi$.

This solution has one discontinuity in each complete period, but when (4.9) is substituted in (3.1) and (3.2) the discontinuity is seen to be one of compression and represents a weak shock wave travelling up and down the tube. For $\omega L/\alpha_0 \approx N\pi$ there are N such shock waves in the tube. In particular at the fundamental resonant frequency there is one shock wave which is repeatedly reflected from the two ends. The solution for $N = 1$ and $r = 0$ is equivalent to that Betchov (1958) constructed by a different argument.

The two solutions (4.4) and (4.9) cover the whole range of r. Figure 3 shows some representative diagrams in the vicinity of resonance, and should be compared with the experimental results shown in figures 1 and 2. For the purpose of comparison, the pressure at the closed end of the tube is proportional to $f(t)$.

Saenger & Hudson (1960) quote a figure of 21·6 cm Hg for the pressure change at first resonance due to the arrival of the shock at the closed end. The present analysis gives 23·5 cmHg; if account is taken of the boundary-layer effect the agreement between theory and experiment is much closer.

One final comment is necessary with reference to the solutions for $|r| < 1$. These have been obtained on the assumption that they are periodic with the same period as the piston oscillations. If this condition is relaxed other solutions are possible. For example, if the piston oscillates at an even resonant frequency ($\omega L/a_0 = N\pi$ with N an even integer), the function

$$f = \epsilon^{\frac{1}{2}} \cos\tau \tag{4.10}$$

satisfies all the required conditions. There seems to be no satisfactory argument at this stage to eliminate such a solution. However, it will appear in the next section that (4.9) does in fact represent the appropriate solution.

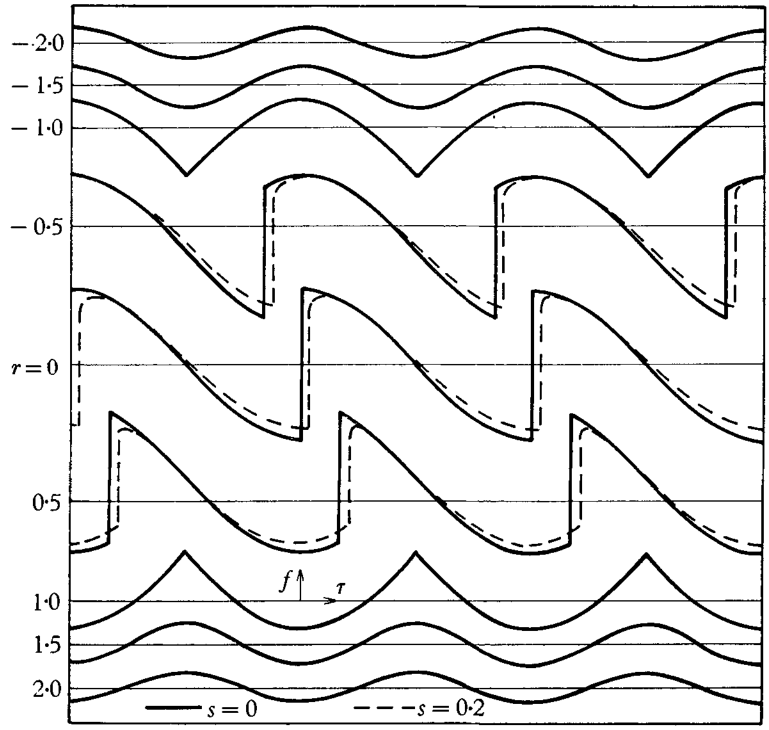

FIGURE 3. The variation of f with τ at various frequencies near resonance. The dotted curves show the modifying effect of the boundary layer when the parameter $s = 0{\cdot}2$.

The piston displacement is in phase with the curves for $r < -1$ near an even resonant frequency ($\cos \omega l/a_0 > 0$) and in phase with the curves for $r > 1$ near an odd resonant frequency ($\cos \omega l/a_0 < 0$). The innermost position of the piston corresponds to a trough on the appropriate curve.

5. Solution with compressive viscosity

The presentation in §4 is included because of its simplicity, but some readers may feel that it relies too much on intuitive reasoning. An argument which is mathematically more satisfactory becomes possible when the effect of compressive viscosity is included.

The equation to be considered is, from (4.2), (4.3) and (3.19) with the constant c suitably redefined,

$$\frac{\delta\omega}{(\gamma+1)\,a_0^2}\frac{df}{d\tau}+(f-2r\epsilon^{\frac{1}{2}}/\pi)^2 = c+\tfrac{1}{2}\epsilon\cos 2\tau. \tag{5.1}$$

This is Ricatti's equation. It can be transformed into a linear equation by the transformation

$$f-2r\epsilon^{\frac{1}{2}}/\pi = \frac{\delta\omega}{(\gamma+1)\,a_0^2}\frac{1}{g}\frac{dg}{d\tau}. \tag{5.2}$$

The function g is then found to satisfy Mathieu's equation

$$\frac{d^2g}{d\tau^2}+(d-2q\cos 2\tau)g = 0, \tag{5.3}$$

where
$$q = \frac{\epsilon}{4}\left(\frac{\delta\omega}{(\gamma+1)a_0^2}\right)^{-2} = (\gamma+1)\frac{l}{L}\left(\frac{\delta\omega}{a_0^2}\right)^{-2} \tag{5.4}$$

and d is some constant still to be determined.

Certain properties of Mathieu functions which are required are discussed below. The reference is McLachlan (1947).

From (5.2) it is clear that, if f is to remain bounded, only those solutions of (5.3) which are never zero need be considered. For a given value of q, and appropriate values of d, there is a class of such solutions which, in general, are of the form

$$e^{\eta\tau}\psi(\tau), \quad e^{-\eta\tau}\psi(-\tau), \tag{5.5}$$

where η is a (real) parameter depending on q and d, and ψ is a periodic function with period π, also depending on q and d. For $\eta \neq 0$, expressions (5.5) represent the two independent solutions of (5.3). When $\eta = 0$ there is one periodic solution, referred to in the literature as $ce_0(\tau, q)$. For a given value of q there is a corresponding value of d, say d_0, which makes $ce_0(\tau, q)$ a solution of (5.3). The class of solutions referred to in (5.5) are all those for which $d \leqslant d_0$. When $d = d_0$ the other solution of (5.3) is not periodic, but is asymptotically proportional to $\tau ce_0(\tau, q)$ for large τ. Thus it gives the same asymptotic representation for f as does $ce_0(\tau, q)$ and we need consider only the latter function.

Solutions of the type (5.5) give a mean value for the right-hand side of (5.2) which is proportional to $\pm\eta$. Thus ce_0 is the appropriate solution of (5.3) to describe the resonant oscillations, since this will give a zero mean value for f when $r = 0$. This case is now considered in detail.

For sufficiently small values of q, ce_0 has the following power series,

$$ce_0 = 1-\tfrac{1}{2}q\cos 2\tau+\tfrac{1}{32}q^2\cos 4\tau-\tfrac{1}{128}q^3(\tfrac{1}{9}\cos 6\tau-7\cos 2\tau)+O(q^4). \tag{5.6}$$

For sufficiently large values of q, the asymptotic behaviour is given by

$$ce_0 \sim \text{const.}\,[W_1\{P_0-P_1\}+W_2\{P_0+P_1\}] \quad (-\tfrac{1}{2}\pi<\tau<\tfrac{1}{2}\pi), \tag{5.7}$$

where
$$\left.\begin{aligned} W_1 &= e^{2q^{\frac{1}{2}}\sin\tau}\frac{\cos(\frac{1}{2}\tau+\frac{1}{4}\pi)}{\cos\tau}, \\ W_2 &= e^{-2q^{\frac{1}{2}}\sin\tau}\frac{\sin(\frac{1}{2}\tau+\frac{1}{4}\pi)}{\cos\tau}, \end{aligned}\right\} \tag{5.8}$$

$$\left.\begin{aligned} P_0 &= 1+\frac{1}{8q^{\frac{1}{2}}\cos^2\tau}+\frac{1}{128q\cos^4\tau}(12-5\cos^2\tau)+\ldots, \\ P_1 &= \frac{\sin\tau}{8q^{\frac{1}{2}}\cos^2\tau}+\frac{\sin\tau}{128q\cos^4\tau}(12+\cos^2\tau)+\ldots, \end{aligned}\right\} \tag{5.9}$$

and the value of the constant is not required in the present context.

Correct to $O(1)$, the above results give

$$\frac{1}{ce_0}\frac{dce_0}{d\tau} = 2q^{\frac{1}{2}}\cos\tau\tanh(2q^{\frac{1}{2}}\sin\tau)\left[1-\frac{1}{4q^{\frac{1}{2}}(1+|\sin\tau|)}-\frac{2\sin(\frac{1}{2}\tau)}{\sinh(4q^{\frac{1}{2}}\sin\tau)}\right] \tag{5.10}$$

and hence, by (5.2),

$$f = \epsilon^{\frac{1}{2}} \cos\tau \tanh(2q^{\frac{1}{2}}\sin\tau)\left[1 - \frac{1}{4q^{\frac{1}{2}}(1+|\sin\tau|)} - \frac{2\sin(\frac{1}{2}\tau)}{\sinh(4q^{\frac{1}{2}}\sin\tau)}\right] \tag{5.11}$$

for $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$, and is periodic with period π.

Note that although (5.7) fails near $\tau = \pm\frac{1}{2}\pi$, expressions (5.10) and (5.11) tend to the (correct) value of zero. A closer inspection of the behaviour of ce_0 in these regions shows that the slope of f, as given by (5.11), is also asymptotically correct and the expression may be used in the closed interval $-\frac{1}{2}\pi \leqslant \tau \leqslant \frac{1}{2}\pi$ without modification.

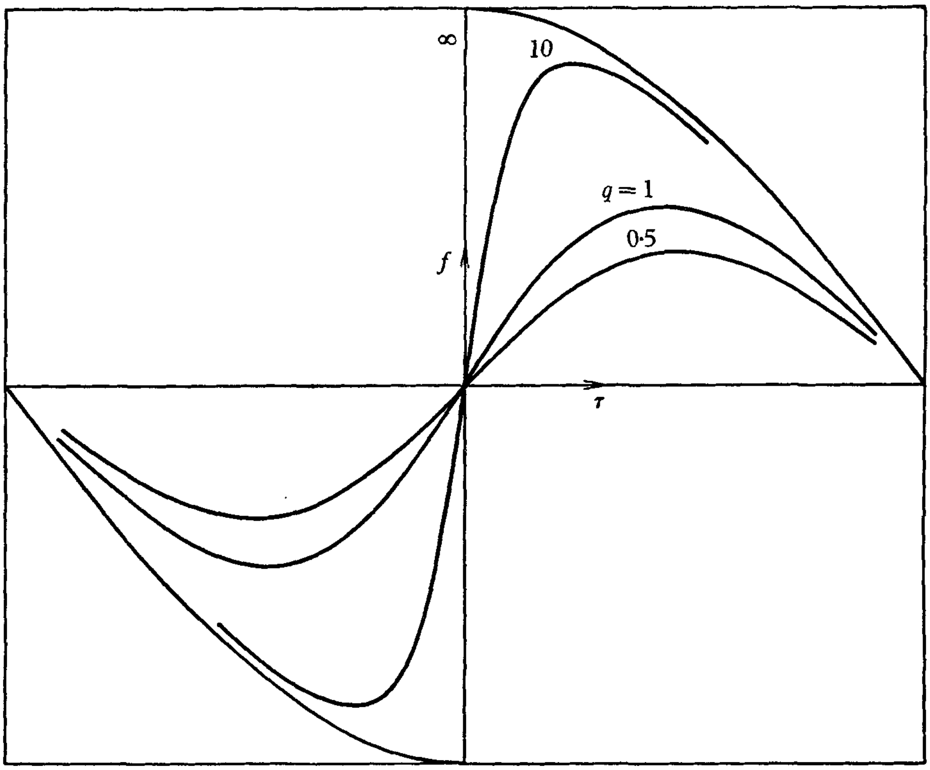

FIGURE 4. The effect of compressive viscosity on the profile of $f(\tau)$ at resonance.

Relation (5.11) represents a simple refinement of (4.9), with $r = 0$, which shows clearly how the actual oscillation jumps from one solution of the inviscid equation to the other.

For intermediate values of q(0(0·05) 2(0·125) 5(0·25) 25) the coefficients in the Fourier series of ce_0 have been tabulated to nine decimal places by the American National Bureau of Standards (1951).

Some feeling for the size of q in a typical experiment can be obtained from the dimensions given by Saenger & Hudson (1960); length of pipe $L = 67$ in., radius of pipe $R = 0{\cdot}95$ in., piston amplitude $l = 0{\cdot}125$ in., first resonant frequency $\omega/2\pi = a_0/2L = 100{\cdot}6$ c/s. From this data one finds that $\epsilon = 3{\cdot}1 \times 10^{-3}$, $\delta\omega/a_0^2 = 1{\cdot}5 \times 10^{-7}$, $q = 2{\cdot}2 \times 10^{11}$ and it is clear that under normal circumstances the effect of compressive viscosity is to produce a very thin shock wave. It is easy to see from (3.1) and (5.11) that the shock wave thickness is of order $Lq^{-\frac{1}{2}}$, which in the above example is 10^{-4} in. Here it would seem reasonable to treat the shock wave as a discontinuity and ignore the structure. However, there may be circumstances in which q takes smaller values, and since the tables produced by the National Bureau of Standards (1951) make calculations over the whole range of q a straightforward matter, figure 4 is included to complete the picture. It shows how the shock wave develops as q increases.

A comparison of the results for $q = 10$ with the approximation

$$f = \epsilon^{\frac{1}{2}} \cos\tau \tanh(2q^{\frac{1}{2}} \sin\tau) \tag{5.12}$$

shows that the maximum error in (5.12) is 12 %. This is reduced to 0·7 % if (5.11) is used.

The solutions near resonance are also periodic with period π in τ (or $2\pi/\omega$ in t) since they can be derived from (5.2) with a function g of the form (5.5) with $\eta \neq 0$. But these Mathieu functions are not tabulated, and since the typical situation is that for which $q \gg 1$, the analysis will now be restricted to this range. Then (4.9) is easily modified to show the effect of viscosity. In (5.1) put $c = \frac{1}{2}\epsilon$ and

$$f = \epsilon^{\frac{1}{2}}\{(2r/\pi) + \phi(\tau)\cos\tau\}. \tag{5.13}$$

The equation then becomes

$$\phi'\cos\tau - \phi\sin\tau = 2q^{\frac{1}{2}}(1-\phi^2)\cos^2\tau. \tag{5.14}$$

In view of the inviscid solution, ϕ is presumably bounded for large q (though not necessarily the derivative of ϕ). Thus (5.14) is approximated to

$$\phi' = 2q^{\frac{1}{2}}(1-\phi^2)\cos\tau, \quad -\tfrac{1}{2}\pi < \tau < \tfrac{1}{2}\pi, \tag{5.15}$$

or

$$\phi = \tanh\{2q^{\frac{1}{2}}(\sin\tau - r)\}. \tag{5.16}$$

Equation (5.13) then gives

$$f = \epsilon^{\frac{1}{2}}\left[\frac{2r}{\pi} + \cos\tau\tanh\{2q^{\frac{1}{2}}(\sin\tau - r)\}\right] \tag{5.17}$$

for $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$ and is periodic with period π. The constant of integration has been chosen so that the mean value of f is zero.

The restriction to the range $-\frac{1}{2}\pi < \tau < \frac{1}{2}\pi$ is necessary because of a difficulty analogous to the Stokes phenomenon in complex variable theory. It arises because the right-hand side of equation (5.14) does not majorize the neglected term $\phi\sin\tau$ uniformly in the neighbourhood of $\cos\tau = 0$. The asymptotic behaviour of ϕ thus requires special consideration in order to continue the function through this point. The general theory is better understood from the point of view of linear differential equations (Erdélyi 1956), and a more complete account for the problem of this paper would start from Mathieu's equation (5.3), namely

$$\frac{d^2g}{d\tau^2} + (d - 2q\cos 2\tau)g = 0$$

rather than (5.14). The same difficulty is present in (5.3) because, for $q \gg 1$, the appropriate solutions are such that $d \sim -2q$, and so the order of magnitude of the term multiplying g is decreased in some neighbourhood of $\cos 2\tau = -1$.

This means that special techniques are required in order to find the asymptotic behaviour of g in the neighbourhood of $\cos 2\tau = -1$. A complete and detailed account for Mathieu's equation is given by Langer (1934). In particular, the Langer theory can be used to derive (5.17), and also to show that this expression actually gives the correct value for f at the end points $\tau = \pm\frac{1}{2}\pi$. The details are involved.

The continuous solutions of (5.1) are adequately represented by (4.4), and the effect of compressive viscosity on these solutions will not be considered.

6. The boundary layer effect

It remains to discuss the influence of the boundary layer on the oscillations. For simplicity the term representing compressive viscosity will be omitted and so, with the constant c suitably redefined, equation (3.19) is equivalent to

$$c+\tfrac{1}{2}\epsilon\cos(\omega t-\tfrac{1}{2}j\pi) = \{f-2r\epsilon^{\frac{1}{2}}/\pi\}^2-\frac{2\beta}{\gamma+1}\int_0^\infty f(t-\xi)\,\xi^{-\frac{1}{2}}\,d\xi. \tag{6.1}$$

With the substitutions $\omega t-j\frac{1}{2}\pi = 2\tau$, $f(t) = \bar{f}(\tau)$, equation (6.1) becomes

$$c+\tfrac{1}{2}\epsilon\cos 2\tau = \{\bar{f}-2r\epsilon^{\frac{1}{2}}/\pi\}^2-\frac{2\beta}{\gamma+1}\left(\frac{2}{\omega}\right)^{\frac{1}{2}}\int_0^\infty \bar{f}(\tau-\xi)\,\xi^{-\frac{1}{2}}\,d\xi. \tag{6.2}$$

Henceforth the bar will be omitted from f, and it will be regarded as a function of τ.

Only the asymptotic behaviour of f is required, as $\tau\to\infty$, and this is a periodic function with period π and zero mean value. It can therefore be written in the form

$$f(\tau) = \frac{2}{\pi}\sum_{n=1}^{\infty}\int_0^\pi f(\xi)\cos\{2n(\tau-\xi)\}\,d\xi, \tag{6.3}$$

and when this is substituted in (6.2) the result is

$$c+\tfrac{1}{2}\epsilon\cos 2\tau = \{f-2r\epsilon^{\frac{1}{2}}/\pi\}^2-2s\epsilon^{\frac{1}{2}}\int_0^\pi f(\xi)\,h(\tau-\xi)\,d\xi, \tag{6.4}$$

where

$$s = \frac{2\beta}{\gamma+1}\left(\frac{\pi}{\epsilon\omega}\right)^{\frac{1}{2}}, \tag{6.5}$$

$$\begin{aligned} h(\tau) &= \frac{1}{\pi}\sum_{n=1}^{\infty} n^{-\frac{1}{2}}\cos(2n\tau-\tfrac{1}{4}\pi) \\ &= \frac{1}{2^{\frac{1}{2}}\pi^{\frac{3}{2}}}\int_0^\infty \frac{\cos 2\tau+\sin 2\tau-e^{-\xi^2}}{\cosh\xi^2-\cos 2\tau}\,d\xi. \end{aligned} \tag{6.6}$$

Note that, although the range of integration for the integral appearing in (6.2) is really finite, it can in fact be replaced by the range $(0<\xi<\infty)$ for the purpose of evaluating the asymptotic behaviour of the integral. For in spite of the presentation of the basic equations as if one were dealing with an initial value problem, once a periodic disturbance is established in the tube the initial conditions become irrelevant. It may then be more illuminating to regard the integral appearing in (6.4) as the additive effect of the boundary layer from each of the components of the spectrum of f.

The series for h follows at once after the substitution of (6.3) in (6.2) and an integration. The integral representation of h can be deduced from the relation

$$\frac{1}{\pi}\sum_{n=1}^{\infty} n^{-\frac{1}{2}} \exp(2in\tau - i\pi/4) = \frac{2}{\pi^{\frac{3}{2}}}\sum_{n=1}^{\infty}\int_0^{\infty} \exp(-n\xi^2+2in\tau-i\pi/4)\,d\xi$$

$$= \frac{2}{\pi^{\frac{3}{2}}}\int_0^{\infty}\frac{\exp(-\xi^2+2i\tau+i\pi/4)}{1-\exp(-\xi^2+2i\tau)}\,d\xi. \tag{6.7}$$

The real part of (6.7) is the expression for h shown in (6.6).

The series representations (6.3) for f and (6.6) for h have no constant term because the mean value of f is zero.

Equation (6.4) is a non-linear integral equation with no simple closed form solution. However, for $s \ll 1$, the solution approximates to the inviscid solution, and this is known to contain discontinuities for $|r| < 1$. On the other hand, for $s \gg 1$, the non-linear term in (6.4) becomes unimportant and the equation approximates to

$$\tfrac{1}{2}\epsilon\cos 2\tau = -4\pi^{-1}r\epsilon^{\frac{1}{2}}f - 2s\epsilon^{\frac{1}{2}}\int_0^{\pi} f(\xi)\,h(\tau-\xi)\,d\xi \tag{6.8}$$

(the constant terms are not required). The solution is

$$f = \frac{-\epsilon^{\frac{1}{2}}\{(8r/\pi)\cos 2\tau + 2s\cos(2\tau+\frac{1}{4}\pi)\}}{\{(8r/\pi)+2^{\frac{1}{2}}s\}^2+2s^2} \tag{6.9}$$

and this expression is continuous for all values of r. Unlike the effect of compressive viscosity, a term of the type introduced into the equation by the boundary-layer effect is not likely to smooth out a discontinuity for sufficiently small values of s. Whether or not there is a finite critical value of s, above which no discontinuities appear, remains an open question.

The data referring to the experiments of Saenger & Hudson give a value of $s = 0{\cdot}25$. For such a value, some idea of the effect of the boundary layer can be obtained by treating it as a small correction to the inviscid solution. However, since at a given resonant frequency s is proportional to the length of the pipe and inversely proportional to the radius, it would not be difficult to produce values of s of order unity.

To solve (6.4) for small values of s and $|r| < 1$, the first approximation is the inviscid solution (4.9). The next approximation will then be the solution of the equation

$$c + \tfrac{1}{2}\epsilon\cos 2\tau + 2s\epsilon\int_{\theta_1}^{\pi+\theta_1}\cos\xi\,h(\tau-\xi)\,d\xi = \{f - 2r\epsilon^{\frac{1}{2}}/\pi\}^2 \tag{6.10}$$

with an appropriate choice of θ_1 and c. The constant c must clearly be chosen so that the left-hand side is never negative. Also, as in the solution of the inviscid equation, unless c is chosen so that the left-hand side is zero for some value of τ (giving the solution a real branch point), discontinuities of rarefaction as well as compression will appear. This is sufficient to determine c and gives, to an adequate approximation,

$$f = \epsilon^{\frac{1}{2}}\left[(2r/\pi)+\cos\tau+\frac{s}{\cos\tau}\int_{\theta_1}^{\pi+\theta_1}\cos\xi\{h(\tau-\xi)-h(\tfrac{1}{2}\pi-\xi)\}\,d\xi\right] \tag{6.11}$$

for $\theta_1 < \tau < \pi+\theta_1$, and otherwise f is periodic with period π.

The value of θ_1 is now determined by the condition that the mean value of f shall be zero. Hence

$$\sin\theta_1 = r + s\int_{\theta_1}^{\pi+\theta_1}\frac{d\tau}{\cos\tau}\int_{\theta_1}^{\pi+\theta_1}\cos\xi\{h(\tau-\xi)-h(\tfrac{1}{2}\pi-\xi)\}\,d\xi \tag{6.12}$$

and $-\frac{1}{2}\pi \leqslant \theta_1 \leqslant \frac{1}{2}\pi$. Since the solution when $s = 0$ gives $\theta_1 = \theta_0 = \sin^{-1} r$ as the first approximation, it is sufficient to use the equation

$$\sin\theta_1 = r + s\int_{\theta_0}^{\pi+\theta_0}\frac{d\tau}{\cos\tau}\int_{\theta_0}^{\pi+\theta_0}\cos\xi\{h(\tau-\xi)-h(\pi/2-\xi)\}\,d\xi \tag{6.13}$$

for small values of s. The substitution of θ_0 for θ_1 in (6.11) is, however, not recommended. This is because, near $\tau = \theta_1$ where f has a discontinuity representing the shock wave, the integral appearing in (6.11) behaves like $(\tau-\theta_1)^{\frac{1}{2}}$. The variation of such a term is sensitive to a small change in θ_1, and its qualitative effect on the profile of f is too significant to permit the approximation. Physically speaking the algebraic behaviour of the integral represents the immediate effect of the interaction between the shock wave and the boundary layer or, to be more precise, it is the attempt of the present theory to describe this interaction. The approximation is at its best when the mathematics simulates this effect in its rightful place, immediately behind the discontinuity.

When $|r| > 1$, the first approximation is given by (4.4), and the second approximation will satisfy the equation

$$c + \tfrac{1}{2}\epsilon\cos 2\tau - 2s\epsilon\,\mathrm{sgn}\,r\int_0^{\pi}(b_0^2+\cos^2\xi)^{\frac{1}{2}}h(\tau-\xi)\,d\xi = \{f - 2r\epsilon^{\frac{1}{2}}/\pi\}^2, \tag{6.14}$$

where b_0 is given by (4.5) and $\mathrm{sgn}\,r = 1$ if $r > 0$ and -1 if $r < 0$.

A suitable approximation to the solution of (6.14) is

$$f = \epsilon^{\frac{1}{2}}\left[\frac{2r}{\pi} - \mathrm{sgn}\,r(b_1^2+\cos^2\tau)^{\frac{1}{2}} + s(b_0^2+\cos^2\tau)^{-\frac{1}{2}}\int_0^{\pi}(b_0^2+\cos^2\xi)^{\frac{1}{2}}\{h(\tau-\xi)-h(\pi/2-\xi)\}\,d\xi\right], \tag{6.15}$$

where b_1 is chosen so that the mean value of f is zero. The approximation has been written in such a form as to ensure that the correction term remains bounded as $b_0 \to 0$.

Some further details of the calculations used in the evaluation of f are given in the appendix. The results of these calculations are shown by the dotted curves in figure 3 for a value of $s = 0{\cdot}2$. In the solutions with a discontinuity the general effect of the boundary layer is a lag in phase and a reduction of the amplitude. The algebraic behaviour of the term in (6.11) representing the boundary-layer effect, referred to above, also softens the profile immediately behind the shock, though the discontinuity is not completely eliminated. An interesting property of this term is that, although it is continuous and periodic with period π, the algebraic behaviour at $\tau = \theta_1+$ is not reproduced as τ tends to $(\theta_1+\pi)$ from below. The reason is that both the symmetric and anti-symmetric parts of this term behave like $(\tau-\theta_1)^{\frac{1}{2}}$ near $\tau = \theta_1$. They augment each other as $\tau \to \theta_1$ from above, but cancel as $\tau \to (\theta_1+\pi)$ from below. There is thus no marked effect on the upstream side of the shock, apart from the reduced amplitude.

For $|r| > 1$ the boundary-layer effect is too small to be represented in figure 3, and is shown on a larger scale in figure 5. The main effect is an increase of amplitude below resonance ($r < 0$) and a decrease above resonance ($r > 0$).

The boundary-layer modification when $|r| = 1$ is not shown because the approximate equations used for the other values of r did not seem to be adequate. It was clear from the calculations that the main effect is near the cusp of the inviscid profile, as one would expect. In this neighbourhood the second approximation is most sensitive to the choice of the first approximation, and to iterate on

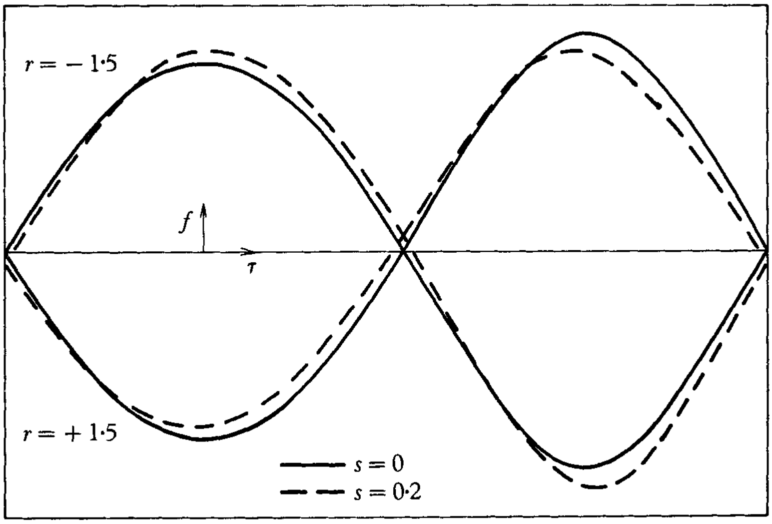

FIGURE 5. The effect of the boundary layer on the continuous oscillations near resonance ($s = 0{\cdot}2$).

the solution for $s = 0$ did not seem sufficiently accurate, at least for $s = 0{\cdot}2$. These cases may well yield to more detailed calculations based on the original equation (6.4), though perhaps the information obtained from the present calculations may be considered sufficient.

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Appendix

In the calculation of f from equation (6.11), namely,

$$\epsilon^{-\frac{1}{2}} f = (2r/\pi) + \cos\tau + \frac{s}{\cos\tau}\int_{\theta_1}^{\pi+\theta_1} \cos\xi\,\{h(\tau-\xi) - h(\tfrac{1}{2}\pi - \xi)\}\,d\xi,$$

it is convenient to write

$$\int_{\theta_1}^{\pi+\theta_1} \cos\xi\, h(\tau-\xi)\,d\xi = (2/\pi)\sin\theta_1\, C(\tau-\theta_1) + (4/\pi)\cos\theta_1\, S(\tau-\theta_1), \quad \text{(A 1)}$$

where $$C(\tau) = \sum_{n=1}^{\infty} \frac{\cos(2n\tau - \frac{1}{4}\pi)}{n^{\frac{1}{2}}(4n^2-1)}, \quad S(\tau) = \sum_{n=1}^{\infty} \frac{n^{\frac{1}{2}}\sin(2n\tau - \frac{1}{4}\pi)}{4n^2-1}. \quad \text{(A 2)}$$

Then

$$\epsilon^{-\frac{1}{2}}f = \frac{2r}{\pi} + \cos\tau + \frac{2s}{\pi\cos\tau}$$
$$\times[\sin\theta_1\{C(\tau-\theta_1) - C(\tfrac{1}{2}\pi-\theta_1)\} + 2\cos\theta_1\{S(\tau-\theta_1) - S(\tfrac{1}{2}\pi-\theta_1)\}] \quad \text{(A 3)}$$

for $\theta_1 < \tau < \pi+\theta_1$, where by (6·12)

$$\sin\theta_1 = r + \frac{s}{\pi}\int_0^\pi [\sin\theta_0\{C(\xi) - C(\tfrac{1}{2}\pi-\theta_0)\}$$
$$+ 2\cos\theta_0\{S(\xi) - S(\tfrac{1}{2}\pi-\theta_0)\}]\frac{d\xi}{\cos(\xi+\theta_0)}, \quad \text{(A 4)}$$

$$\sin\theta_0 = r, \quad \text{(A 5)}$$

with $$-\tfrac{1}{2}\pi \leqslant \theta_0 \leqslant \tfrac{1}{2}\pi, \quad -\tfrac{1}{2}\pi \leqslant \theta_1 \leqslant \tfrac{1}{2}\pi.$$

The functions $C(\tau)$, $S(\tau)$ were calculated by summation of the series. These are slowly convergent, but the truncation error is easily evaluated from the asymptotic equality

$$\sum_N^\infty \frac{n^{\frac{1}{2}}e^{2ni\tau}}{4n^2-1} \sim \frac{1}{\pi^{\frac{1}{2}}}e^{(2N-1)i\tau}\int_0^\infty \frac{e^{-(N-\frac{1}{2})\xi^2}}{\sinh(\frac{1}{2}\xi^2 - i\tau)}\{\xi^2 + \tfrac{1}{15}\xi^6 + \tfrac{1}{945}\xi^{10} + \ldots\}\,d\xi. \quad \text{(A 6)}$$

The following table shows the results of the calculation. Note that

$$S \sim -0{\cdot}52663 + (\pi\tau/2)^{\frac{1}{2}} \text{ near } \tau = 0.$$

τ/π	C	S	τ/π	C	S
0	0·29863	−0·52663	0·55	−0·26188	0·13528
0·05	0·36888	−0·08587	0·60	−0·29442	0·07043
0·10	0·37056	0·06286	0·65	−0·30537	−0·00171
0·15	0·33475	0·15886	0·70	−0·29280	−0·07900
0·20	0·27407	0·22298	0·75	−0·25543	−0·15920
0·25	0·19722	0·26260	0·80	−0·19270	−0·24008
0·30	0·11124	0·28150	0·85	−0·10477	−0·31934
0·35	0·02227	0·28198	0·90	0·00754	−0·39487
0·40	−0·06420	0·26587	0·95	0·14271	−0·46457
0·45	−0·14322	0·23489	1·00	0·29863	−0·52663
0·50	−0·21040	0·19075			

When $|r| > 1$, the appropriate equation for f is (6.15), namely,

$$\epsilon^{-\frac{1}{2}}f = \frac{2r}{\pi} - \operatorname{sgn} r(b_1^2 + \cos^2\tau)^{\frac{1}{2}}$$
$$+ s(b_0^2+\cos^2\tau)^{-\frac{1}{2}}\int_0^\pi (b_0^2+\cos^2\xi)^{\frac{1}{2}}\{h(\tau-\xi) - h(\tfrac{1}{2}\pi-\xi)\}\,d\xi.$$

If we define $$J_n = \int_0^{\frac{1}{2}\pi}\frac{\cos 2n\xi}{(b_0^2+\sin^2\xi)^{\frac{1}{2}}}d\xi \quad \text{(A 7)}$$

then (6.15) can be written

$$\epsilon^{-\frac{1}{2}}f = \frac{2r}{\pi} - \operatorname{sgn} r\,(b_1^2+\cos^2\tau)^{\frac{1}{2}} - s(b_0^2+\cos^2\tau)^{-\frac{1}{2}}B(\tau), \quad \text{(A 8)}$$

where $$B(\tau) = \frac{1}{4\pi}\sum_{n=1}^{\infty} n^{-\frac{3}{2}}\{\cos(2n\tau - n\pi - \tfrac{1}{4}\pi) - \cos\tfrac{1}{4}\pi\}\{J_{n-1} - J_{n+1}\} \tag{A 9}$$

and the constants b_0 and b_1 are determined from the relations

$$\int_0^{\frac{1}{2}\pi} (b_0^2 + \cos^2\tau)^{\frac{1}{2}}\,d\tau = (1+b_0^2)^{\frac{1}{2}}E\{(1+b_0^2)^{-\frac{1}{2}}\} = |r|, \tag{A 10}$$

$$\int_0^{\frac{1}{2}\pi} (b_1^2 + \cos^2\tau)^{\frac{1}{2}}\,d\tau = (1+b_1^2)^{\frac{1}{2}}E\{(1+b_1^2)^{-\frac{1}{2}}\}$$
$$= |r| + \frac{s\,\mathrm{sgn}\,r}{2^{\frac{5}{2}}\pi}\sum_{n=1}^{\infty} n^{-\frac{3}{2}}(J_0 - J_n)(J_{n-1} - J_{n+1}). \tag{A 11}$$

Equation (A 7) for J_n is unsuitable for computation when n is large, but the integral can be transformed to give

$$J_n = -\frac{i}{2}\int \frac{z^{n-1+\frac{1}{2}}\,dz}{(\cot^2\frac{1}{2}\alpha - z)^{\frac{1}{2}}(z - \tan^2\frac{1}{2}\alpha)^{\frac{1}{2}}}, \tag{A 12}$$

where $\cot\alpha = b_0$ and the integration is taken round the unit circle in the complex z-plane. The integrand has two branch points inside the unit circle, at $z = 0$ and $z = \tan^2(\frac{1}{2}\alpha)$. Thus the path of integration can be deformed into the two sides of the branch cut from $z = 0$ to $z = \tan^2(\frac{1}{2}\alpha)$. This gives

$$J_n = \int_0^{\tan^2\frac{1}{2}\alpha} \frac{x^{n-1+\frac{1}{2}}\,dx}{(\cot^2\frac{1}{2}\alpha - x)^{\frac{1}{2}}(\tan^2\frac{1}{2}\alpha - x)^{\frac{1}{2}}} = \int_0^{\frac{1}{2}\pi} \frac{2(\tan\frac{1}{2}\alpha)^{2n-1}\cos^{2n}\xi\,d\xi}{(\cot^4\frac{1}{2}\alpha - \cos^2\xi)^{\frac{1}{2}}}. \tag{A 13}$$

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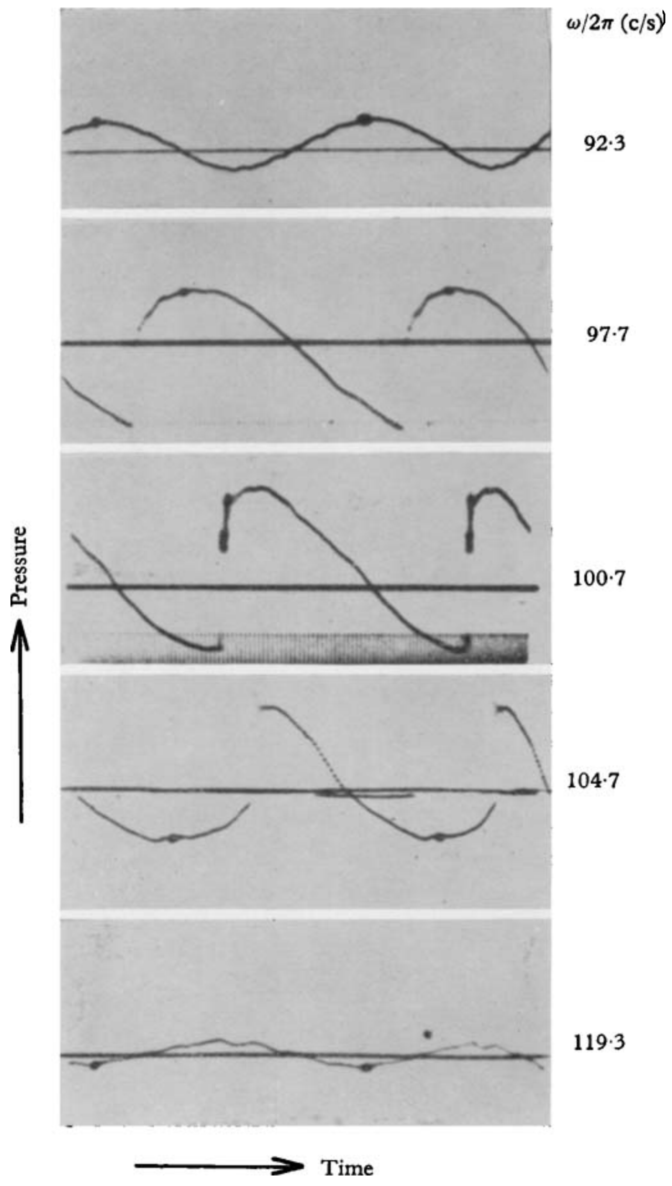

FIGURE 1. Pressure variation observed at the closed end of a 1·9 in. diameter tube of length 67 in. for frequencies near the fundamental frequency of 100·6 c/s. The piston amplitude is 0·125 in. Whenever a dot appears on the records, the piston is at its inmost position in the tube. (Reproduced with the permission of Dr R. A. Saenger.)

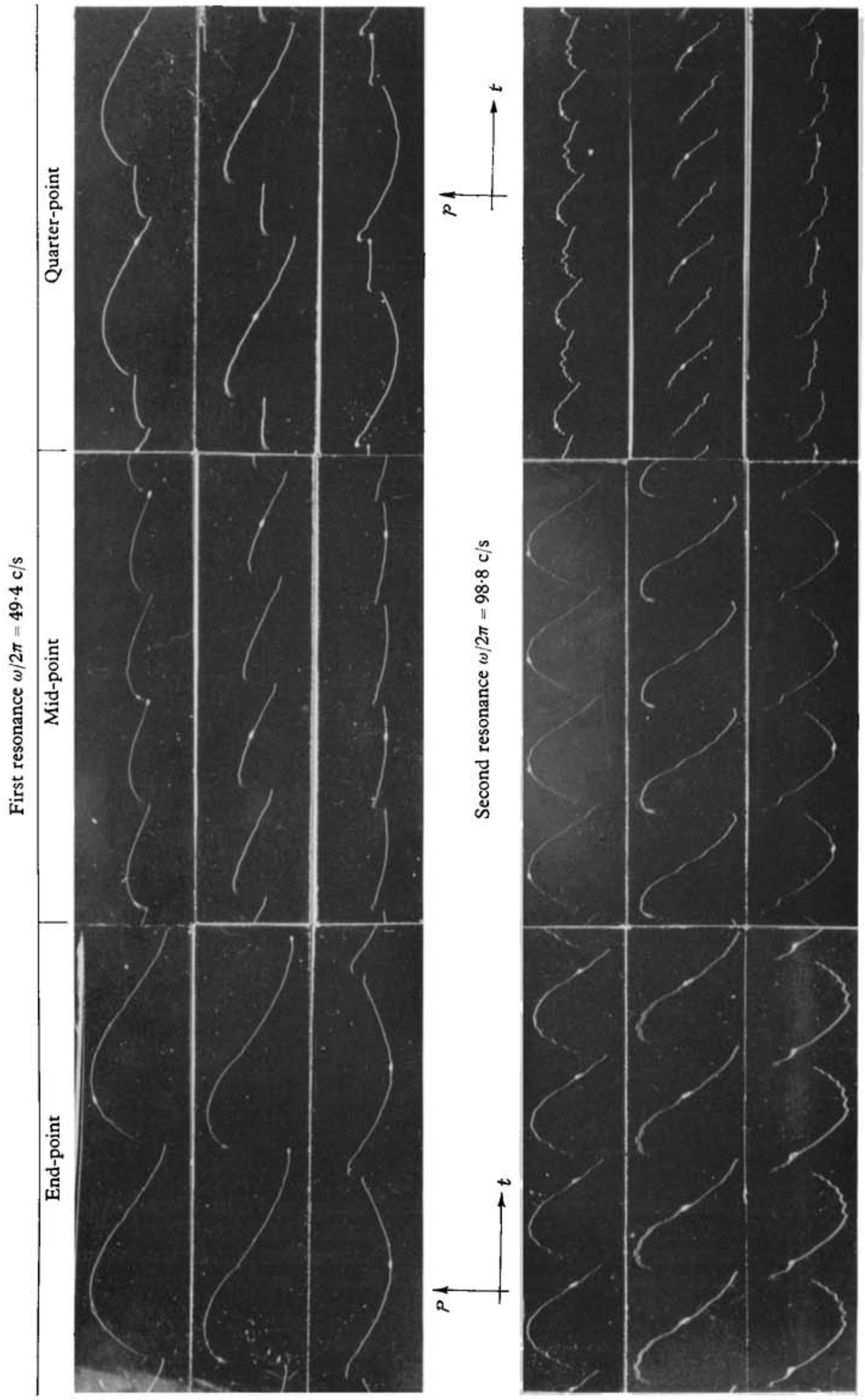

FIGURE 2. Pressure variation at various positions along a tube for near resonant frequencies (diameter 1·9 in., length 137 in., piston amplitude 0·125 in.). (Reproduced with the permission of Dr R. A. Saenger.)